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THE NINE-POINTS CIRCLE.

By MR. R. D. BOHANNAN, University of Virginia.

ABC is a triangle. A_a is the foot of the altitude from A. A_m is the middle point of the side opposite A. The altitudes intersect at O. A_0 is the middle point of AO.

From similar triangles BCC_a, BAA_a,

$$\begin{aligned} BA_a : BC_a &:: BA : BC \\ &:: BC_m : BA_m; \\ \therefore BA_a . BA_m &= BC_a . BC_m, \end{aligned}$$

and therefore A_a, A_m, C_a, C_m, lie on a circle.

From similar triangles COA_a, CBC_a,

$$\begin{aligned} CO : CA_a &:: CB : CB_a; \\ \therefore CC_0 : CA_a &:: CA_m : CC_a; \\ \therefore CC_0 . CC_a &= CA_a . CA_m; \end{aligned}$$

$\therefore$ C_0 lies on the circle A_a, A_m, C_a,

that is, A_a, A_m, C_a, C_m, C_0 lie on a circle;
Similarly, A_a, A_m, C_a, C_m, A_0 “ “ “ “
$\therefore$ A_a, A_m, C_a, C_m, A_0, C_0 “ “ “ “
Similarly, A_a, A_m, B_a, B_m, A_0, B_0 “ “ “ “
These two circles have three points in common;

$\therefore$ A_a, A_m, A_0, B_a, B_m, B_0, C_a, C_m, C_0

lie all on the same circle.

SOLUTIONS OF EXERCISES.

12

The result

$$-\frac{p^2q^2 + 4p^3r - 8q^3 + 2pqr + 9r^3}{(r - pq)^2}$$

is given as the equivalent of the function

$$\left(\frac{\beta - \gamma}{\beta + \gamma}\right)^2 + \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2,$$

where α, β, γ are the roots of the cubic

$$x^3 + px^2 + qx + r = 0.$$

Is this result correct? [*A. Hall.*]

SOLUTION.

The result is not correct, for if so then it must be true when $\beta = 0$, in which case $r = 0$, and the two expressions reduce to

$$\frac{8q}{p^2} - 1, \text{ and } \left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)^2 + 2.$$

Now if these are identical, since $p^2 = (\gamma + \alpha)^2$ and $p^2 - 4q = (\gamma - \alpha)^2$, we have

$$\frac{8q}{p^2} - 1 = \frac{p^2 - 4q}{p^2} + 2;$$

or

$$3q = p^2,$$

which is not generally true. [*J. W. Nicholson.*]

14

A THIN inextensible cord in which the density of the material increases in geometric progression, as the distance from one end increases in arithmetic progression is laid directly across a rough horizontal cylinder the circumference of a normal section of which is equal to twice the length of the cord; to determine the co-efficient of friction, supposing the cord to be only just supported when its extremities are in the horizontal plane containing the axes of the cylinder. [*William Hoover.*]

SOLUTION.

Let O be the centre of a right section, APB, of the cylinder. The string lies along APB, having its ends at A and B, the smaller end being at A. Let a be the radius of the cylinder. Call the density at A, unity. By the conditions of the problem, the density at P is $e^{a\theta}$, θ being the angle AOP. Consider the equilibrium of an element, PQ, of the string. The string is about to move in the direction of PQB. Let T be the tension at P, $T + dT$ that at Q, R the normal reaction at P, and μ the co-efficient of friction. Resolving the forces acting on the element, PQ, along the tangent and normal at P, we have

$$dT - \mu Ra\,d\theta - age^{a\theta}\cos\theta\,d\theta = 0,$$
$$T - Ra\,d\theta - age^{a\theta}\sin\theta\,d\theta = 0.$$

Eliminating R, we have

$$\frac{dT}{d\theta} - \mu T = age^{a\theta}(\mu \sin\theta + \cos\theta),$$

a linear differential equation of the first order and degree. Multiplying by $e^{-\mu\theta}$

$$\frac{d}{d\theta}(Te^{-\mu\theta}) = age^{(a-\mu)\theta}(\mu \sin\theta + \cos\theta);$$

$$\therefore T = age^{\mu\theta}\int e^{(a-\mu)\theta}(\mu\sin\theta + \cos\theta)\,d\theta$$

$$= age^{\mu\theta}\left\{ C + e^{(a-\mu)\theta}\frac{(\mu(a-\mu)+1)\sin\theta + (a-2\mu)\cos\theta}{1+(a-\mu)^2}\right\}.$$

$T = 0$, when $\theta = 0$; $\quad \therefore C = \dfrac{2\mu - a}{1+(a-\mu)^2}$

And $T = 0$, when $\theta = \pi$;

$$\therefore (2\mu - a) + e^{(a-\mu)\pi}(2\mu - a) = 0;$$

$$\therefore (2\mu - a) = 0, \text{ or } 1 + e^{(a-\mu)\pi} = 0.$$

The second equation is not satisfied by any real value of μ. The first equation gives $\mu = \frac{a}{2}$. [*R. D. Bohannan.*]

14.

A FAISCEAU of parabolas can be drawn having the pole of a cardioid as the common focus, all passing through one point and all cutting the cardioid at right angles. [*H. A. Newton.*]

SOLUTION.

Take a faisceau of straight lines, all passing through the point, (ai), in the plane of the complex variable $Z = x + yi$; also a faisceau of circles having the point, (ai), as common centre.

The equation of any one of the straight lines is

$$y = mx + a. \tag{1}$$

Being transformed by the function of the second degree,

$$w = Z^2, \text{ or } w + vi = (x + yi)^2 = x^2 - y^2 + 2xyi,$$

whence

$$w = x^2 - y^2,$$
$$v = 2xy,$$

$$x^2 = \frac{\pm\sqrt{w^2+v^2}+w}{2},$$

$$y^2 = \frac{\pm\sqrt{w^2+v^2}-w}{2};$$

equation (1) becomes

$$4m^2w^2 + 4(m^3 - m)wv + (m^4 - 2m^2 + 1)v^2 + 4a^2(m^2 - 1)w - 8a^2mv - 4a^4 = 0. \qquad (2)$$

As m varies from tan 0° to tan π, this equation represents a system of parabolas, all having the origin as focus and all passing through the point, $(w = -a^2)$.

The equation of any one of the circles in the variable plane is

$$x^2 + (y-a)^2 = r^2.$$

Being transformed by the same function, $w = Z^2$, this becomes

$$\sqrt{w^2+v^2} = a\sqrt{2\left(\sqrt{w^2+v^2}-w\right)} + r^2 - a^2,$$

which reduces to

$$[w^2 + 2a^2w + v^2 + (a^2 - r^2)^2]^2 = 4r^4(w^2+v^2). \qquad (3)$$

Since the circles and straight lines with which we set out intersect at right angles, and since in the transformation employed similarity in the smallest parts is maintained; therefore, as r varies, equation (3) represents a faisceau of curves, all of which cut the parabolas defined by (2) at right angles. Thus by this transformation a faisceau of parabolas is obtained having a common focus, all passing through one point and all cutting at right angles the curves defined by equation (3). All of these latter curves lie symmetrically about the axis of w.

When $r > a$, the curve is of the fourth order and forms a loop around the origin.

When $r < a$, it is still of fourth order, but does not form a loop.

When $r = \frac{a}{\sqrt{2}}$, the curve touches the imaginary axis at two points and lies wholly on the negative side of that axis.

When $r = a$, equation (3) becomes

$$(w^2 + 2a^2w + v^2)^2 = 4a^4(w^2+v^2),$$

which is that of a cardioid, having the origin or common focus of the parabolas for its pole. This is the special case required by the problem.

[*A. G. Reeves.*]

15

IT IS required to inscribe a rectangle of breadth nD within another rectangle whose diagonal is B. Show that the excess E of the declivity M of the diagonal of the greater rectangle over that of the length of the less may be calculated when n is small, from the formula

$$\sin E = n \cos 2M[1 + 2n \sin 2M + n^2(3 - \cos 2M) + \ . \ . \ . \ ].$$

[*William M. Thornton.*]

SOLUTION.

If m denote the declivity of the length of the inscribed triangle, we have two right angles whose corresponding sides

$$D \cos M - nD \sin m, \text{ and } nD \cos m\,;$$
$$D \sin M - nD \cos m, \text{ and } nD \sin m\,;$$

are proportional. Therefore

$$\frac{\cos M - n \sin m}{\sin M - n \cos m} = \frac{\cos m}{\sin m},$$

whence $$\sin (M - m) = n \cos 2m,$$

or $$\sin E = n \cos 2 (M - E),$$

or $$\sin E = n \cos 2M(1 + 2 \tan 2M \sin E - 2 \sin^2 E - 4 \tan 2M \sin E \sin^2 \tfrac{1}{2}E).$$

Accordingly we have to terms of the first order in n

$$\sin E_1 = n \cos 2M;$$

to terms of the second order

$$\sin E_2 = n \cos 2M(1 + 2n \sin 2M);$$

to terms of the third order

$$\sin E_3 = n \cos 2M[1 + 2n \sin 2M + n^2 (1 - 3 \cos 4M)],$$

and so on.

This is the well known problem of the design of the angle blocks for the Howe bridge-truss. [*W. M. Thornton.*]

20

A TUNNEL section, consisting of a rectangle surmounted by a semicircle, is required to accommodate a rectangle of breadth B and height H. Show how to determine its proportion so that its (1) area, (2) perimeter shall be least.

[*William M. Thornton.*]

SOLUTION.

If x denote the radius of the circular top, we shall find

1. For the perimeter

$$P = 2H + (\pi + 2)x - \sqrt{4x^2 - B^2};$$
$$\therefore \frac{dP}{dx} = \pi + 2 - \frac{4x}{\sqrt{4x^2 - B^2}};$$

whence for a minimum value of P

$$x = \frac{B}{2} \cdot \frac{\pi + 2}{\sqrt{\pi(\pi + 4)}}.$$

2. For the area

$$A = 2Hx + \tfrac{1}{2}\pi x^2 - x\sqrt{4x^2 - B^2};$$
$$\therefore \frac{dA}{dx} = 2H + \pi x - \frac{8x^2 - B^2}{\sqrt{4x^2 - B^2}} = 0$$

for a minimum value of A. This gives rise to a quartic which is solved most easily by trial.

Railway rolling stock of ordinary dimensions will require on a double-tracked road, with tracks $13'$ between centres, a rectangle $22'$ broad and $14'$ high above the bottom of the rails. Adding $1'$ for clearance we have

$$B = 24, \qquad H = 15;$$
$$\therefore x_1 = 13.326, \quad x_2 = 13.534;$$

corresponding to tunnel sections $26'.65 \times 22'.53$ and $27'.07 \times 22'.28$ respectively. To the second dimension the requisite addition for ballast and drain must, of course, be added. [*W. M. Thornton.*]

21

IF THE central force on a body moving in a parabola, latus rectum $4m$, were to cease acting at the vertex and continue interrupted till the body had described an angle of 60° about the focus; determine the orbit it would afterwards describe. [*William Hoover.*]

SOLUTION.

We have for the point where the force again begins to act, the well known relations

$$\frac{V^2}{k^2} = \frac{p}{r^2 \sin^2 \phi} = \frac{2}{r} - \frac{1}{a} \qquad (a)$$
$$= \frac{2}{r_0} = \frac{2}{m}, \text{ by hypothesis,}$$

where V is the velocity of the body in its orbit, k the constant of the system, ψ the angle between the tangent to the new orbit and the prolongation of the corresponding radius vector (r), r_0 the radius vector for the vertex of the old orbit, and p, a, and e are the semi-parameter, semi-major axis, and eccentricity of the new orbit. For the given point we have, as may easily be seen,

$$\begin{aligned} r &= 2m, \\ \psi &= 30^\circ, \\ r^2 \sin^2 \psi &= m^2; \end{aligned}$$

whence, by substitution in (a), we see that the new orbit will be an hyperbola, in which $p = 2m$, $a = -m$, $e = \sqrt{3}$. [*Ormond Stone.*]

EXERCISES.

26

DETERMINE the maximum right cone B inscribed in a given right cone A, the vertex of B being at the centre of the base of A. [*O. Root, Jr.*]

27

THE NUMBER of points common to three surfaces of the mth, nth, and pth degrees being in general mnp, find the co-ordinates of all the real and imaginary points of intersection of the three surfaces, $y^2 + x^2 + z^2 = 4x$, $x^2 = yz$, and $y^2 = x^4$. [*H. A. Newton.*]

28

GIVEN the perpendicular, median, and bisector issuing from one and the same vertex of a plane triangle and terminating in the opposite side, to construct the triangle and determine a formula for its area. [*Marcus Baker.*]

29

GIVEN a pair of points A,B, if C,D are such that $OD \,.\, OC = OA^2 = OB^2$ and $AOC = AOD$, prove that the pair A,B bears similar relation to the pair C,D. Show, also, the existence of a pair E,F which bears the same relation to each of the pairs A,B and C,D. [*Wm. Woolsey Johnson.*]